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SHORT COMMUNICATIONS

A Constructive Proof of Kirszbraun's Theorem

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The subject matter of this paper is Kirszbraun's theorem, which says that any 1-Lipschitz (nonexpansive) map from a subset of $\mathbb{E}^d$ to $\mathbb{E}^d$ can be extended to the entire space $\mathbb{E}^d$. Valentine [1] proved a similar assertion for spherical and hyperbolic spaces. Here and in what follows, $\mathscr{X}$ denotes one of the three spaces $\mathbb{E}^d$, $\mathbb{S}^d$, and $\mathbb{H}^d$.

Theorem (Kirszbraun, Valentine [1]). *Let U be a subset of $\mathscr{X}$. Then any nonexpansive map $f\colon U \to \mathscr{X}$ can be extended to a nonexpansive map $f'\colon \mathscr{X} \to \mathscr{X}$.*

There are a number of interesting generalizations of Kirszbraun's theorem; see, e.g., [2]–[5].

All existing proofs of Kirszbraun's theorems are analytic. Danzer, Grunbaum, and Klee [6] posed the problem of constructing a simple geometric extension.

In this paper, we show how to construct an extension of a function f in the class of PL-isometries (that is, length-preserving piecewise linear maps) in the case of a finite set U. Let us give a more rigorous definition.

Definition. A *PL-isometry* $\varphi\colon P \to \mathscr{X}$ of a polytope P in $\mathscr{X}$ is defined as a continuous map for which there exists a locally finite triangulation of P such that, for any simplex T from this triangulation, the restriction of φ to T is a motion.

A visual example of a PL-isometry is the map transforming a usual sheet of paper into an origami figure.

Main Theorem . *Any weakly contractive map f on a finite set of points $U \subset \mathscr{X}$ can be extended to a PL-isometry on the entire space.*

Remark. Using Helly's theorem and the fact that the space $\mathscr{X}$ is always separable, we can easily show that the main theorem implies the Kirszbraun–Valentine theorem in full generality (not only for finite U).

Clearly, PL-isometries are nonexpansive (such maps are also called short or weakly contractive), i.e., do not increase distances between pairs of points.

Suppose that a PL-isometry φ takes points A and B to $\varphi(A)$ and $\varphi(B)$. Suppose also that there exists a motion g for which

$$\varphi(A) = g(A) \qquad \text{and} \qquad \varphi(B) = g(B).$$

Then, by virtue of weak contraction, we have $\varphi(X) = g(X)$ for any point X of the interval $[A, B]$ (otherwise, the triangle inequality would be violated).

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Therefore, each motion from the set of motions in a PL-isometry corresponds to one domain, and this domain is convex. In what follows, we refer to such domains as *sheets* and assume that all sheets correspond to different motions.

Consider two sheets L and L' neighboring along a hyperface. The motions of the sheets coincide on this hyperface, but they are different; thus, the motion of the first sheet is the composition of the motion of the second sheet and the symmetry with respect to the hyperface (because there exists only two motions coinciding in a hyperplane), i.e.,

$$g_{L'} = g_L \circ \operatorname{sym}(L \cap L').$$

Below we mention yet another important property, which is a convenient tool for constructing various PL-isometries and is useful for proving the main theorem.

Lemma. *For any PL-isometry φ on a finite polytope P, there exist a finite set of points such that these points and their images under φ uniquely determine the PL-isometry φ.*

Proof. In the case of $\mathbb{E}^d$ and $\mathbb{H}^d$, it suffices to take all vertices of all polyhedra. Since each convex polyhedron is the convex hull of its vertices, it follows that all sheets (which are convex polyhedra in the case under consideration) can be uniquely reconstructed.

This property can be generalized to the entire spherical space. But such a generalization involves a difficulty, because a sheet may contain hemispheres as faces, and the sheet vertices are not sufficient for reconstructing them (e.g., $\mathbb{S}^2$ contains a whole family of semicircles with common endpoints). Let us cut the sphere $\mathbb{S}^d$ by $d+1$ pairwise perpendicular hyperplanes. These hyperplanes cut also the sheets of the PL-isometry under consideration. The new small sheets cannot contain hemispheres; therefore, the action of the PL-isometry on them is determined by the action on the vertices. $\square$

Proof of the main theorem. If the dimension of the space under consideration is 0, then the assertion is obviously true. Further, we use the assertion of the theorem for $\mathbb{S}^{d-1}$.

We argue by induction on the number of points at which the map f is given.

If the map is given at only one point, then the required assertion is obvious. Indeed, the entire space can be considered to be a single sheet, and the corresponding motion is any motion taking the given point $A \in U$ to $f(A)$.

Thus, suppose that the theorem is proved in the case where the map is defined at no more than $n-1$ points. Let us prove it for the case in which the number of points is n.

We denote the points of the set U by $A_1, A_2, \dots, A_n$. Without loss of generality, we can assume that A_n is a fixed point of f. By the induction hypothesis, there exists a PL-isometry ψ (defined on the entire d-space) which takes A_i to B_i for $i = 1, \dots, n-1$.

Suppose that A_n is not a fixed point of ψ (otherwise, we can take ψ for the required PL-isometry). Consider the set of points X moving away from A_n under ψ, or, in other words, of points X for which

$$d(A_n, X) < d(A_n, \psi(X)).$$

We denote this set by Ω. Clearly, Ω is nonempty and open (because ψ is a continuous map and the inequality determining Ω is strict), and it does not fill the entire space, because the points A_i with $i = 1, \dots, n-1$ do not belong to it (they do not move away from A_n by the assumption of the theorem).

Let us show that Ω contains the entire interval $[A_n, X]$ together with any point X. Take any point Y on this interval. Since the map ψ is weakly contractive, it follows that

$$d(X, Y) \geq d(\psi(X), \psi(Y)) \qquad \text{and} \qquad d(A_n, X) < d(A_n, \psi(X)),$$

which implies

$$d(A_n, Y) = d(A_n, X) - d(X, Y) < d(A_n, \psi(X)) - d(\psi(X), \psi(Y)) \leq d(A_n, \psi(Y)).$$

The last inequality and $d(A_n, X) < d(A_n, \psi(X))$, follows from the triangle inequality.

Consider how the boundary of Ω is formed. Let L be a sheet of the PL-isometry ψ, and let g_L be the corresponding motion. Consider the point $g_L^{-1}(A_n)$ (it does not necessarily belong to L). It is easy to see that $L \cap \Omega$ is the set of points L which are nearer to A_n than to $g_L^{-1}(A_n)$. If this set is not empty and does not coincide with the entire sheet L (in these cases, L is contained entirely outside or inside Ω, which is of no interest for us), then the part of the boundary of Ω contained in L is simply the intersection of L with the middle hyperplane between the points A_n and $g_L^{-1}(A_n)$. Thus, considering all sheets of ψ, we see that the boundary of Ω consists of convex polytopes.

Now let us describe the construction of the required map φ. Outside the set Ω, φ coincides with ψ. Let L' be the part of the boundary of Ω contained inside the sheet L. Consider the pyramid with vertex at A_n and base L' (or, simpler, the convex hull of these point and set); we take it for the sheet of φ'. To this pyramid we assign the motion $g_L \circ s_{L'}$, where $s_{L'}$ is the symmetry with respect to the hyperplane containing L'. By the construction of this symmetry, we have

$$s_{L'}(A_n) = g_L^{-1}(A_n).$$

We perform this procedure for the entire boundary Ω.

Suppose that a point X belongs to L'. It is easy to see that the motion $g_L \circ s_{L'}$ takes the interval $[X, A_n]$ to $[\psi(X), A_n]$. Indeed, we have

$$g_L \circ s_{L'}(X) = g_L(X) = \psi(X) \qquad \text{and} \qquad g_L \circ s_{L'}(A_n) = g_L(g_L^{-1}(A_n)) = A_n,$$

and the interval $[X, A_n]$ is contained inside one sheet of the map ψ.

Thus, on the boundary of two sheets, the corresponding motions coincide, because they coincide on the boundary of Ω.

If we have defined φ on the entire set Ω, then we have obtained the required map. However, φ may remain undefined on some part of Ω. Consider the ray going away from A_n. If it intersects the boundary Ω, then the map φ is defined on it, but if it is contained in Ω, then A_n is the only point on the ray at which the map φ is defined.

Let us show that, in the case of $\mathbb{S}^d$, the map φ is already defined on the entire space. Note that the point A_n' opposite to A_n cannot move away from A_n (because its distance from A_n is already maximal). Let us show that it belongs to the interior of the complement to Ω.

1. If A_n' is not fixed, then it approaches A_n; by continuity, it approaches A_n together with one of its neighborhoods.

2. If $\psi(A_n') = A_n'$, then it is easy to see that the point A_n' has a neighborhood such that

$$A_n'X = A_n'\psi(X)$$

for any point X from this neighborhood (for such a neighborhood we can take all points of the sheets containing A_n'). Since the points A_n and A_n' are opposite, it follows that, for any point of the sphere, we have

$$d(A_n, X) = d(A_n, A_n') - d(A_n', X) = d(A_n, \psi(X)).$$

Therefore, the chosen neighborhood does not intersect Ω. Thus, any arc of the great circle $[A_n, A_n']$ intersects the boundary Ω, and hence the map φ is defined on it.

Now let us show that φ can also be extended in the case of the spaces $\mathbb{E}^d$ and $\mathbb{H}^d$.

Consider a sphere S centered at A_n and small enough to be contained inside Ω. The incomplete PL-isometry φ constructed above ("incomplete" means that its domain does not coincide with the entire space) maps S to itself. Thus, we have an incomplete PL-isometry on S. It follows from the lemma that the motion on each sheet can be uniquely reconstructed from the motion of several points (the vertices and some interior and boundary points of the sheet). We mark all such points for each sheet. If we have an PL-isometry acting as φ on all of these points, then it coincides with φ on the sphere S. Such a PL-isometry exists, because the theorem is valid for spherical spaces. This isometry can be extended to the entire set Ω. Indeed, if a point X on the sphere is mapped to a point X', then we translate the ray $[A_n, X)$ (to be more precise, its part inside Ω) to $[A_n, X')$. The map thus defined coincides with φ at the points at which φ is defined. Therefore, extending φ in this way, we obtain a PL-isometry acting on the entire space. $\square$

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REFERENCES

1. F. A. Valentine, Amer. J. Math. **67** (1), 83 (1945).
2. U. Lang and V. Schroeder, Geom. Funct. Anal. **7** (3), 535 (1997).
3. B. Grünbaum, Proc. Amer. Math. Soc. **13** (5), 812 (1962).
4. K. Ball, Geom. Funct. Anal. **2** (2), 137 (1992).
5. W. Johnson and J. Lindenstrauss, in *Contemp. Math.*, Vol. 26: *Conference in Modern Analysis and Probability, New Haven, Conn., 1982* (Amer. Math. Soc., Providence, R. I., 1984), Vol. 26, pp. 189–206.
6. L. Danzer, B. Grünbaum, and V. Klee, *Helly's Theorem and Its Relatives* (Amer. Math. Soc., Providence, RI, 1963; Mir, Moscow, 1968).